Symmetric Balanced Incomplete Block Designs (Sbibd) And Construction of Hadamard Rhotrices

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Abstract. Balanced Incomplete Block Designs (BIBD) are the designs obtained from the arrangement of symbols in blocks following established rules and satisfying relevant necessary conditions. BIBDs have broad applications in many fields.The entries in a Hadamard matrix H are all +1's or -1's, satisfying the property $HH^T = n\ I_n$ [2]. Rhotrix is a new concept in mathematics having applications in cryptography. In this paper, using SBIB designs the Hadamard Rhotrices are constructed. AMS Subject classification: 05B05, 05B20, 62K10.

Key words: SBIB designs, Hadamard matrices, Hadamard Rhotrices, incidence matrices, coupled matrices.

1. Introduction

A design is called a Balanced Incomplete Block (BIB) design if v symbols are arranged in b blocks, each block containing k ($<v$) symbols, every symbol occurs at the most once in a block and in exactly r blocks ensuring that every pair of symbols occurs together in λ blocks. The parameters of the BIBD are v, b, r, k, λ. These parameters satisfy the following necessary conditions.

$$vr = bk$$

$$\lambda(v\text{-}1) = r(k\text{-}1)$$

$$b \geq v \text{ (Fisher's inequality)}$$

A symmetric BIB design is the one in which if $v = b$. In an SBIB design any two blocks have exactly λ common symbols. Also (r- λ) shall be a perfect square when v is even number. R. N. Mohan et. al [1] characterized all the SBIB designs and a generalized parameters has been obtained. The parameters of all the designs have been tabulated for the use of researchers for constructional methods.

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For a BIB design, its incidence matrix is a (v x b) matrix defined by $A = [a_{ij}]$, where

$$a_{ij} = 1 \text{ if } x_i \text{ belongs to } B_j$$

$$0 \text{ if } x_i \text{ is not a member of } B_j$$

Where $V = \{x_1, x_2, \ldots\ldots x_v\}$ are the symbols and $B = \{B_1, B_2, \ldots\ldots B_b)$ are the blocks.

Many researchers have constructed the Hadamard matrices by using different methods [3, 5, 8]. The concept of Rhotrix [4] is introduced in 2003 with elements that are placed in between a second order and third order matrices. Higher order Rhotrix of order 'm' is defined that is coupled with two matrices of order (m-1) and (m-2) respectively [6]. In [7] Hadamard matrix is defined over a finite field.

2. Main result

In this paper, using the parameters of an irreducible Symmetric Balanced Incomplete Block (SBIB) design and that of a special type of SBIBD, Hadamard rhotrices are constructed and these rhotrices can be used extensively in coding theory.

Theorem. 2.1. If D_1 and D_2 are the block designs of two Symmetric Balanced Incomplete Block designs $v = b = 4s+3, r = k = 2s+1, \lambda = s$ *and* $v = b = 4s+2, r = k = 4s+1, \lambda = 4s$ respectively then there exists a Hadamard Rhotrix R of order 8s + 5 whose coupled matrices are of order 4s+3 and 4s+2 respectively.

Proof.

Consider the SBIB designs D_1 and D_2 whose parameters are $v = b = 4s+3, r = k = 2s+1, \lambda = s$ *and* $v = b = 4s+2, r = k = 4s+1, \lambda = 4s$ respectively. Let M_1 and M_2 be the incidence matrices of the SBIBDs D_1, D_2 respectively.

Columns.1 2 3………… 2s+1 2s+2 2s+3........... 4s+3

$$\text{Let } M_1 = \begin{pmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots 1 & 0 & 0 \ldots\ldots 0 \ldots\ldots 0 \\ 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots 1 \ldots\ldots 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 1 \ldots\ldots 1 \ldots\ldots 1 \end{pmatrix}$$

In M_1, each row (column) has 4s+3 symbols. Here in each row (column) there must be (2s+1) 1's and consequently (4s+3) - (2s+1) = (2s+2) 0's, since r=k=2s+1. Hence M_1 is symmetric.

$$M_1M_1^T = \begin{pmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots 1 & 0 & 0 \ldots\ldots 0 \ldots\ldots 0 \\ 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots 1 \ldots\ldots 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 1 \ldots\ldots 1 \ldots\ldots 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots 1 & 0 & 0 \ldots\ldots 0 \ldots\ldots 0 \\ 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots 1 \ldots\ldots 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 1 \ldots\ldots 1 \ldots\ldots 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 2s+1 & s & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots\ldots & s \\ s & 2s+1 & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots\ldots & s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s & s & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots & 2s+1 \end{pmatrix}$$

This is a symmetric matrix of order 4s+3 in which all the diagonal elements are equal to r=k=2s+1 and all other entries are λ = s. Now, (r-λ) I_{4s+3} + λ J = (2s+1-s) I_{4s+3} + t J = (t+1) I_{4s+3} + sJ, where J is an all 1 matrix of order 4s+3.

$$= (s+1) \begin{pmatrix} 1 & 0 & 0 & 0\ldots\ldots\ldots\ldots 0 \\ 0 & 1 & 0 & 0\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 1 & 0\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 0 & 1\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0 & 0\ldots\ldots\ldots\ldots 1 \end{pmatrix} + s \begin{pmatrix} 1 & 1 & 1 & 1\ldots\ldots\ldots\ldots 1 \\ 1 & 1 & 1 & 1\ldots\ldots\ldots\ldots 1 \\ 1 & 1 & 1 & 1\ldots\ldots\ldots\ldots 1 \\ 1 & 1 & 1 & 1\ldots\ldots\ldots\ldots 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & 1 & 1 & 1\ldots\ldots\ldots\ldots 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2s+1 & s & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots\ldots & s \\ s & 2s+1 & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots\ldots & s \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s & s & s\ldots\ldots\ldots s & s & s\ldots\ldots s\ldots & 2s+1 \end{pmatrix} = M_1M_1^T$$

Since $M_1M_1^T$ = (r-λ) I+ λ J, M_1 is the incidence matrix of a SBIBD. The order of M_1 is 4s+3 and this matrix is the first coupled matrix corresponding to the SBIBD with parameters $v = b = 4s+3,\ r = k = 2s+1,\ \lambda = s$ is

Now, the incidence matrix M_2 for D_2 is given by

Columns. 1 2 3………… 4s 4s+1 4s+2

$$M_2 = \begin{pmatrix} 1 & 1 & 1\ldots\ldots\ldots\ldots 1 & 1 & 0 \\ 1 & 0 & 1\ldots\ldots\ldots\ldots 1 & 1 & 1 \\ 1 & 1 & 0\ldots\ldots\ldots\ldots 1 & 1 & 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 1 & 1\ldots\ldots\ldots\ldots 1 & 1 & 1 \end{pmatrix}$$

Then

(i) There are 4s+2 symbols in each row (column) of the matrix.

(ii) The number of blocks is b = 4s+2, hence there are '4s+2' columns (rows) in M_2.

(iii) There are 4s+1, 1's in each row (column) as the size of each block is 4s+1

(iv) Each symbol occurs in r=4s+1blocks.

(v) Each pair of symbols is repeated in λ = 4s blocks.

$$M_2M_2^T = \begin{pmatrix} 1 & 1 & 1................1 & 1 & 0 \\ 1 & 0 & 1................1 & 1 & 1 \\ 1 & 1 & 0................1 & 1 & 1 \\ .... \\ .... \\ 0 & 1 & 1................1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1................1 & 1 & 0 \\ 1 & 0 & 1................1 & 1 & 1 \\ 1 & 1 & 0................1 & 1 & 1 \\ .... \\ .... \\ 0 & 1 & 1................1 & 1 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 4s+1 & 4s & 4s................4s & 4s \\ 4s & 4s+1 & 4s.................4ts & 4s \\ 4s & 4s & 4s+1..............4s & 4s \\ .... \\ .... \\ 4s & 4s & 4s..............4s & 4s+1 \end{pmatrix}$$

We know that if A is the incidence matrix of a BIB design then $AA^T = (r-\lambda) I + \lambda J$ holds good, where J is a matrix of all 1of order that of A.

$$(r-\lambda) I + \lambda J = (4s+1-4s) I + 4sJ$$

$$= I + 4sJ$$

$$= \begin{pmatrix} 4s+1 & 4s & 4s................4s & 4s \\ 4s & 4s+1 & 4s.................4ts & 4s \\ 4s & 4s & 4s+1..............4s & 4s \\ .... \\ .... \\ 4s & 4s & 4s..............4s & 4s+1 \end{pmatrix} = M_2M_2^T$$

Thus M_2, the incidence matrix of the SBIB design shall be considered as the second coupled matrix of the Rhotrix. The coupled matrices M_1 and M_2 obtained from the SBIB designs can be used to construct the Hadamard Rhotrix of order (4s+3) + (4s+2) = (8s+5).

Illustration. 2.1.

Let the SBIBD be *v = b = 4s+3, r = k = 2s+1, λ = s. Let s*= 1. Therefore v=b=7, r=k= 3, λ = 1

The design D_1 is given by

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

The incidence matrix of D_1 is

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

For M_2, consider the SBIBD with parameters $v = b = 4s+2,\ r = k = 4s+1,\ \lambda = 4s$. Let s=1. So that v=b= 6, r=k= 5, $\lambda = 4$

The design D_2 = {(1 2 3 4 5), (1 3 4 5 6), (1 2 4 5 6), (1 2 3 4 6), (1 2 3 5 6), (2 3 4 5 6)}

Therefore, the incidence matrix M_2 of D_2 is given by

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

With M_1 and M_2 as the coupled matrices, the Rhotrix R_{13} is constructed as follows.

$$R_{13} = \left\langle \begin{array}{c} 1 \\ 1 \quad 1 \quad 1 \\ 1 \quad 1 \quad 0 \quad 1 \quad 1 \\ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \\ 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \\ 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \\ 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \\ 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \\ 1 \quad 1 \quad 0 \quad 1 \quad 1 \\ 1 \quad 1 \quad 1 \\ 0 \end{array} \right\rangle$$

Illustration 2.2.

Consider the SBIB design with parameters $v = b = 4s+3,\ r = k = 2s+1,\ \lambda = s$. *Let s=2.* Therefore v=b=11, r=k= 5, λ = 2. The design is {(1 2 3 4 5), (1 2 6 7 8), (1 3 6 9 11), (1 4 7 9 10), (1 5 8 10 11), (2 3 8 9 10), (2 4 6 10 11), (2 5 7 9 11), (3 4 7 8 11), (3 5 6 7 10), (4 5 6 8 9)}

whose incidence matrix is

$$M_1 = \begin{pmatrix} 1&1&1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&1&1&1&0&0&0\\ 1&0&0&1&0&0&1&0&1&1&0\\ 1&0&1&0&0&1&0&0&1&0&1\\ 1&0&0&0&1&0&0&1&0&1&1\\ 0&1&0&1&0&1&0&0&0&1&1\\ 0&1&1&0&0&0&0&1&1&1&0\\ 0&1&0&0&1&0&1&0&1&0&1\\ 0&0&1&1&0&0&1&1&0&0&1\\ 0&0&1&0&1&1&1&0&0&1&0\\ 0&0&0&1&1&1&0&1&1&0&0 \end{pmatrix}$$

Obtain the Matrix M_2 for the SBIB design $v = b = 4s+2,\ r = k = 4s+1,\ \lambda = 4s$. Let s=2. So that v=b= 10, r=k= 9, λ = 8

Corresponding (10, 9, 8) design is {(1 2 3 4 5 6 7 8 9), (1 3 4 5 6 7 8 9 10), (1 2 4 5 6 7 8 9 10), (1 2 3 5 6 7 8 9 10), (1 2 3 4 6 7 8 9 10), (1 2 3 4 5 7 8 9 10), (1 2 3 4 5 6 8 9 10), (1 2 3 4 5 6 7 9 10), (1 2 3 4 5 6 7 8 10). (2 3 4 5 6 7 8 9 10)}

$$\text{Incidence matrix } M_2 = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&0\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&1&1\\ 1&1&1&1&1&0&1&1&1&1\\ 1&1&1&1&1&1&0&1&1&1\\ 1&1&1&1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1&1&0&1\\ 0&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

Using M_1 and M_2 as the coupled matrices of order 11 and 10 respectively the following Hadamard rhotrix is constructed whose order is 21.

R_{21} =

```
                              1
                            1 1 1
                          1 1 1 1 1
                        1 1 0 0 0 1 1
                      1 1 0 1 0 1 0 1 1
                    0 1 0 1 1 0 1 1 0 1 0
                  0 1 1 1 0 1 0 1 0 1 1 1 0
                0 1 1 1 0 1 0 0 0 1 0 1 1 1 0
              0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
            0 1 0 1 0 1 0 1 0 0 0 1 0 1 0 1 0 1 0
          0 0 0 1 1 1 0 1 0 1 1 1 0 1 0 1 1 1 0 0 0
            0 1 1 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 0
              0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0
                1 1 1 1 0 1 1 0 1 1 0 1 1 1 1
                  1 1 1 1 1 1 0 1 1 1 1 1 1
                    1 1 1 1 1 0 1 1 1 1 1
                      0 1 0 1 0 1 0 1 0
                        1 1 0 0 0 1 1
                          1 1 1 1 1
                            0 1 0
                              0
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3. Conclusion

Hadamard Rhotrices can be extensively used in coding theory and as it is a new field of study, there is much scope to explore more properties of the rhotrices and thereby construction of more efficient codes

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